An accurate formula for the period of a simple pendulum oscillating beyond the small-angle regime

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ABSTRACT

A simple approximation formula is derived here for the dependence of the period of a simple pendulum on amplitude that only requires a pocket calculator and furnishes an error of less than 0.25% with respect to the exact period. It is shown that this formula describes the increase of the pendulum period with amplitude better than other simple formulas found in literature. A good agreement with experimental data for a low air-resistance pendulum is also verified and it suggests, together with the current availability/precision of timers and detectors, that the proposed formula is useful for extending the pendulum experiment beyond the usual small-angle oscillations.

I. INTRODUCTION

The periodic motion exhibited by a simple pendulum is harmonic only for small-angle oscillations, for which there is a well-known period formula.[1] Beyond this limit, the equation of motion is nonlinear, which makes difficult the mathematical description of the oscillations.[2] Although an integral formula exists for the period of such nonlinear system, valid for any amplitude,[2] it is often avoided in introductory physics classes because it is not possible to evaluate such integral exactly by applying the Fundamental Theorem of Calculus. This is why almost all introductory physics textbooks and lab manuals recommend the readers to restrict the study to small-angle oscillations, for which the approximation $\sin\theta \cong \theta$, with θ in radians, works and a harmonic motion is obtained. The pedagogical advantage is that the linearized equation has a simple exact solution, whose derivation is promptly understood by first-year students.[1] At this point, the authors would like to emphasize that such linearization has bothered them since their own undergraduate times because the amplitude needs to be less than 7^{o} if an error below 0.1% (the typical experimental error obtained with a stopwatch) is desired and the reader should recognize that pendulum applications with such small oscillations are rare.[3] Indeed, as the authors and their colleagues have noted, the more interested students often ask for a formula that could describe the pendulum period for oscillations beyond the small-angle regime wishing to explore the motion for larger amplitudes and then to compare its period to that for small amplitudes. In fact, the restriction to small-angle oscillations hinders the understanding of real-world behavior since the isochronism observed in this regime soon vanishes for increasing amplitudes.[4] From the experimental viewpoint, this is also unnecessary because a millisecond precision in period measurements is easily obtained with current technology (accurate timers and detectors).[5–8] For instance, an experimental error of the order of 0.1% or less is typically obtained with a one meter-long pendulum, a fact that gives support to accurate experimental studies of the dependence of the period on amplitude for large-angle oscillations, even in introductory physics labs.[7,8] However, such experiments have not been encouraged by the instructors and it should be due to the difficulty in finding a simple but accurate analytical formula for the pendulum period, i.e. a formula that only requires a few operations on a pocket calculator and whose deviations from the exact values are of the same order of the experimental error.

In this paper, a closed-form approximation formula for the pendulum period with the features pointed above is proposed. Comparisons to similar attempts published recently, as well as to experimental data gathered from literature and taken by us, are also given.

II. APPROXIMATION SCHEME

A particle of mass m suspended by a massless rigid rod of length L that is fixed at the upper end, moving in a vertical circle, composes an *ideal* simple pendulum, which oscillates with a symmetric restoring force (in the absence of dissipative forces) due to the force of gravity.[2] This "simple" mechanical system is illustrated in Fig. 1 and its equation of motion can be obtained by equating the gravitational torque to the product of the moment of inertia and the angular acceleration (see, e.g., Refs. 1–3). The resulting differential equation for the angular displacement simplifies to

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0 \ , \tag{1}$$

where g is the local acceleration of gravity. Note that we chose $\theta=0$ at the stable equilibrium position (see the vertical dashed line in Fig. 1). For a given initial condition, i.e. once $\theta(0)$ and $d\theta/dt(0)$ are chosen, the exact solution for the initial value problem can be obtained only numerically. Therefore, some approximation should always be assumed in searching for an analytical formula for the pendulum period, which is much desired in view to cover this topic in introductory physics classes. For small-angle oscillations, the approximation $\sin\theta \cong \theta$ is valid and Eq. (1) becomes a linear differential equation analogue to that for the simple harmonic oscillator. Within this regime, the pendulum oscillates harmonically with a period that tends to $T_0 = 2\pi\sqrt{L/g}$ as the amplitude tends to zero, a well-known textbook formula.[1] As will be discussed further, T_0 underestimates the exact period for any amplitude, but this is almost imperceptible in the small-angle regime wherein the oscillations are practically *isochronous*, since T_0 does not depend on amplitude. Beyond the small-angle regime, T_0 becomes unsuitable and Eq. (1) can be taken up again for a direct numerical solution. On the other hand, an integral expression for the exact pendulum period may be derived based only upon energy considerations, without detailed discussions on differential equations. Since the system is conservative, the principle of conservation of mechanical energy applies and will be used to put the velocity as a function of θ. Taking the zero of potential energy at the lowest point on the path of the pendulum bob, as seen in Fig. 1, and choosing for simplicity the initial conditions as $\theta(0) = \theta_0$ and $d\theta/dt(0) = 0$, one has[2]

$$mgL(1-\cos\theta_0) = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) . \tag{2}$$

This equation may be solved for $d\theta/dt$, which results in

$$\frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{L}\left(\cos\theta - \cos\theta_0\right)} \ , \tag{3}$$

where the + (–) sign is for the counter-clockwise (clockwise) motion, according to Fig. 1. Integrating $d\theta/dt$ for the motion from θ_0 to 0 (thus taking the "–" sign into account) and noting that such displacement requires a time interval equal to a quarter of *T*, the *exact* pendulum period, one has

$$T = 2\sqrt{2}\sqrt{\frac{L}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \ . \tag{4}$$

This definite integral cannot be solved in a closed-form in terms of elementary functions (i.e., the integrand has not an antiderivative), which is a feature common to all elliptical integrals.[2,10] Indeed, the evaluation of the exact pendulum period through Eq. (4) by applying numerical integration techniques is not straightforward because there is a vertical asymptote to the function $1/\sqrt{\cos\theta - \cos\theta_0}$ at $\theta = \theta_0$, which makes the integral *improper*. Therefore, the usual Newton-Cotes rules for numerical integration do not furnish accurate results, as pointed out by Schery (who applied Simpson's rule).[11] Fortunately, this difficulty can be circumvented by substituting $\cos\theta$ by $1 - 2\sin^2(\theta/2)$ and then making a change of variable, given implicitly by $\sin\varphi = \dfrac{\sin(\theta/2)}{\sin(\theta_0/2)}$. This changes Eq. (4) to

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \ , \tag{5}$$

where $k \equiv \sin(\theta_0/2)$. The above definite integral is $K(k)$, the *complete elliptic integral of the first kind*, which is not improper since $k < 1$ for any $\theta_0 < \pi$ rad. Thus, it is not difficult to evaluate T with the aid of a computer for a given amplitude, since standard numerical integration codes are largely available in many programming languages (e.g., FORTRAN, PASCAL, C, etc.). However, this task can become much tedious if only a pocket calculator is available, as usually occurs in introductory physics classes.

For a comparison with T_0, the small-angle period approximation, it is more appropriate to write T as $T_0 \times \dfrac{2}{\pi} K(k)$. This leads to the following relative error:

$$RE_0 = \frac{\pi}{2K(k)} - 1 \ , \tag{6}$$

which depends on θ_0 implicitly through k. As may be verified graphically, the absolute value of this error increases rapidly with θ_0.[9]

The approximation formula for the pendulum period that is being proposed here in this paper comes from the perception that $f(\varphi;k) \equiv \sqrt{1-k^2\sin^2\varphi}$, i.e. the denominator of the integrand in $K(k)$, is a smooth function for $0 \le \varphi \le \pi/2$ (i.e., the limits of integration). This is true for any θ_0 between 0 and $\pi/2$ rad,[12] which corresponds to k between 0 and $\sqrt{2}/2$, as shown in Fig. 2. Taking the points (0,1) and $(\pi/2, a)$ for a linear interpolation, where $a \equiv f(\pi/2;k) = \sqrt{1-k^2} = \cos(\theta_0/2)$, it is found that

$$r(\varphi;\theta_0) = 1 - \frac{2}{\pi}(1-a)\varphi \tag{7}$$

is the straight line that can be taken for approximating $f(\varphi;k)$ in the range $0 \le \varphi \le \pi/2$. An approximation, then, arises for $K(k)$:

$$K(k) \cong \int_0^{\pi/2} \frac{d\varphi}{1-\frac{2}{\pi}(1-a)\varphi} = -\frac{\pi}{2}\frac{\ln(a)}{1-a} \; . \tag{8}$$

This leads to a simple approximation formula for the exact period:

$$T_{\log} = -2\pi\sqrt{\frac{L}{g}}\,\frac{\ln(a)}{1-a} \; . \tag{9}$$

Note that $\ln(a) < 0$, hence $T_{\log}$ is positive. Despite the simplicity of this formula, it is important to check out its accuracy in representing the exact pendulum period. This task is simplified if one writes $T_{\log}$ as $-T_0 \frac{\ln(a)}{1-a}$, which furnishes an error $RE_{\log} = -\frac{\pi}{2K(k)}\frac{\ln(a)}{1-a} - 1$.

In the next section, the accuracy of the approximation formula established in Eq. (9) in representing the exact pendulum period, as given by Eq. (5), will be compared to that of other approximation formulas found in literature.

III. COMPARISON WITH OTHER APPROXIMATIONS

The accuracy of the logarithmic approximation proposed in Eq. (9), above, for the large-angle pendulum period should now be compared to that of other approximation formulas found in the physics teaching literature, for amplitudes below $\pi/2$ rad.[12]

The errors found in approximating T, given in Eq. (5), by T_0 and $T_{\log}$, as well as by other approximation formulas are depicted in Fig. 3. Clearly, the small-angle approximation, whose relative error is RE_0, exhibits the worst behavior since its error becomes greater than 0.1% (0.5%) for amplitudes above 7° (16°). Perhaps the most famous formula for the large-angle period is the approximation given by Bernoulli from a perturbative analysis of Eq. (5). Truncating the resulting series at the 2nd term, one obtains

$$T_2 = T_0 \left(1 + \frac{\theta_0^{\ 2}}{16} \right) . \tag{10}$$

It was a surprise to us to verify that this formula (in fact, one of the most used) leads to an error that increases rapidly, overcoming 0.1% (0.5%) for amplitudes above 41° (60°), as seen observing the curve RE_2 in Fig. 3. Therefore, it is inadequate for studying large-angle pendulum periods. One may even argue that the addition of more terms improves the accuracy of T_2, but all terms up to and including the 8th-order one should then be included (see Ref. 14) and it makes the formula both voluminous and unpractical.

More recently, other approximation formulas for the pendulum period were proposed. Among these formulas, the Kidd-Fogg one has attracted much interest due to its simplicity.[8] It is given by:

$$T_{KF} = T_0 \frac{1}{\sqrt{\cos\left(\frac{\theta_0}{2}\right)}} . \tag{11}$$

The dash-dotted line in Fig. 3 represents the error committed by assuming T_{KF} as the exact period. Clearly, it furnishes an error greater than 0.1% only for amplitudes above 57°, reaching 0.8% for 90°. Then, it is only reasonable for interpreting the experimental data taken for the pendulum period in the large-angle regime, contrarily to what is pointed out by Millet,[15] who argues that it should be included in textbooks.

Another approach for creating an approximation formula for the large-angle period is to make an interpolatory-like linearization directly in Eq. (1), as first done by Molina.[16] The resulting expression is simply

$$T_M = T_0 \left(\frac{\sin\theta_0}{\theta_0} \right)^{-3/8} , \tag{12}$$

which furnishes an error greater than 0.1% only for amplitudes above 69° (see the thin solid curve in Fig. 3). Although it seems to be acceptable, the error curve reaches ~0.4% for θ_0 = 90°, thus it is not so good for much large amplitudes.

At last, the error curve for the logarithmic formula we are proposing here for the pendulum period, represented by $RE_{\log}$ in Fig. 3, remains below all other error curves for any amplitude below 90°. Note that it is above 0.1% only for amplitudes greater than 74°. Moreover, it increases slowly, reaching only 0.2% for an amplitude of 86°. This shows that our formula works well even for very large amplitudes (near 90°). In other words, $T_{\log}$ approximates the exact period better than other simple formulas found in literature.

IV. EXPERIMENT AND RESULTS

For checking the applicability of the new approximation formula developed here for the period of a simple pendulum oscillating beyond the small-angle regime a comparison to reliable experimental data is required. In fact, this is mandatory since we are intending to furnish a formula for helping students to interpret their own experimental data for large-angle pendulum periods.

Unfortunately, accurate experimental data for the dependence of the period of a simple pendulum on amplitude are not abundant in the physics teaching literature. It seems that the more reliable ones are the data collected by Fulcher and Davis (see Ref. 4) using a pendulum made with a piano wire and observing two complete runs and the data published by Curtis (see Ref. 17) in which the period was taken as the average for ten successive periods, for each amplitude. Both works are good examples of accurate period measurements made with an ordinary stopwatch. Of course, the measurement of the time interval for n successive periods is a good strategy for oscillations in the small-angle regime, where all the runs spend almost the same time, but not for large-angle oscillations because the period decreases considerably from one oscillation to the next due to air damping, leading to an average period shorter than the desired period of the first oscillation (for which the amplitude is just the measured initial angle θ_0). Clearly, both set of data contain such underestimation for the period of large-angle oscillations, as may be seen in Fig. 4, where the ratio T/T_0 is plotted as a function of the amplitude (the typical graph requested from students in lab manuals). Also included in this figure is the set of measurements taken by us in a more sophisticated experiment whose arrangement details were the subject of a recent paper.[5] In our experiment, both the time-keeping and position detection processes were done automatically in a manner to reduce the experimental error in the period measurements to the μs scale (note that the error in time-keeping when a common stopwatch is used is of the order of 0.2 s, i.e. the human reaction time). Indeed, we decided to measure the period by keeping only the time interval between two successive passages over the lower point of the pendulum's circular path (i.e., $T/2$) in order to reduce damping effects on the measured period, mainly for large amplitudes.

It is important to mention that we devoted much attention to the reduction of air resistance to the motion of the pendulum's bob. This was done by implementing an electronic process for time-keeping and position detection, as mentioned above, and also by choosing suitable materials and parameters for the simple pendulum. In this way, since the small size and large weight of the pendulum's bob is an important factor for reducing the effect of air resistance on the pendulum period, we choose lead as the bob's material due to its higher density in comparison to other metals. This allowed for a body that is both small and weighty (m = 0.400 kg). We also found that the cylindrical form is preferable over the spherical one for it allows a better localization of the center of mass, which in needed for measuring L accurately. Another advantage is the possibility of reducing the damping by reducing the scattering cross-section of air, i.e. by choosing a diameter much smaller than the height of the cylinder, which led us to fabricate a body that we called a "pen of lead". For this massive pendulum we verified that cords made of *nylon*, possibly the most used material, are inadequate since they stretch considerably throughout each oscillation, particularly for large-angle oscillations, and it causes undesired vibrations. The more convenient material, taking into account lightness (see Ref. 18 for the importance of this factor), price, and availability, seems to be cotton, thus we used a common sewing thread as the pendulum cord. We also investigated the length of the pendulum's cord that furnishes the better results for large-angle oscillations. After comparing many lengths for an amplitude of 60^o, we choose a length of 1.500 m in view to circumvent the difficulties related to the damping in fast oscillations obtained with shorter cords, since the air resistance increases with velocity.[19] Additionally, this length accounts for a small (say, not tedious) period of about 2.5 s.

All these precautions led us to much accurate experimental data for the pendulum period, for amplitudes between 0 and 90^o, as may be confirmed in Fig. 4 where it is easy to see that our experimental data (black points) are more near the exact period expected in the absence of air resistance (the solid line) than the data published in Refs. 4 (crosses) and 17 (circles). Also in Fig. 4, it is clear that our logarithmic approximation formula is in better agreement to the experimental data than the other simple formulas found in literature.

Of course, we developed a pendulum experiment whose precision goes beyond that of a typical one worked out in introductory physics labs, but this was essential for obtaining a reliable set of experimental data for the pendulum period as a function of amplitude. Moreover, with the increase of the presence of sophisticated electronic equipments and computers in introductory physics labs it will not be difficult to instructors to advise their students on developing an experiment similar to the one we carried out.

V. CONCLUSIONS

In this paper, a simple approximation formula relating the pendulum period to the amplitude whose accuracy is better than all other simple formulas found in literature is proposed and tested experimentally. The closed-form approximate expression that arises when a linear interpolation is made for the integrand of the elliptical integral that appears in the exact period expression only requires a pocket calculator for period evaluations and furnishes an error of less than 0.25% when compared to the exact period (found numerically). The other formulas found in literature most consists on deriving corrections to the small-angle approximation by taking either a Maclaurin polynomial approximation for $\sin\theta$ directly in the equation of motion[2,11,20] or by applying perturbation theory,[4,21] but these approaches are not simple for first year students. The logarithmic formula proposed here was also tested experimentally, presenting a better agreement with the data measured with low air-resistance pendulums. For reducing the effect of air resistance, which is usually the main source of experimental error in large-angle pendulum experiments,[5,8,17] we choose the pendulum material and format carefully and also adopted an automatic process for accurate time interval measurements. Moreover, the usual strategy of keeping the time interval corresponding to many successive oscillations and then taking the average period as the experimental value of the period of the first run was verified to be inadequate for measuring the period of large-angle oscillations because the amplitude and the period itself decay rapidly from one oscillation to the next, which leads to averages that are smaller than the true period for the first run. This inconvenience was overcome by measuring only a half of the period in the first run, for each amplitude. These strategies lead to accurate experimental data that may be used for comparisons to the existing approximation formulas for the pendulum period as a function of amplitude. This compares favorably to our approximation formula, which is in better agreement to experimental data taken by us and gathered from literature. In closing, the measurement of the period of a simple pendulum – a standard activity in introductory physics labs – could become more interesting for students if teachers extend the period measurements to large-angle oscillations and adopt the logarithmic formula proposed here. The spontaneous classroom discussions we watched in our own classes during and after the large-angle pendulum experiment, mainly on the manner the period increases with amplitude and how it could be measured, motivated us to exchange this experience with other teachers (the readers) in viewing to divulgate our approach to this old theme.

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be analyzed here instead of that with respect to T_0.

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FIGURE CAPTIONS

Fig. 1. The simple pendulum circular motion. The pendulum bob is released from a position that forms an angle θ_0 with the vertical, at rest, and passes at an arbitrary position $\theta\,(<\theta_0)$ with a velocity $L\;d\theta/dt$. Note that the height of the mass *m* depends on θ according to $L-L\,\cos\theta$.

Fig. 2. Behavior of the function $f(\varphi;k)=\sqrt{1-k^2\sin^2\varphi}$ for φ between 0 and $\pi/2$ rad and for some values of $k(\theta_0)$. The horizontal and vertical dashed lines are $f(\varphi;k)=1$ and $\varphi=\pi/2$ rad, respectively. The dash-dotted line is the linear interpolation curve $r(\varphi;\theta_0)$ for $\theta_0=\pi/2$ rad.

Fig. 3. Comparison of the relative errors committed by using the approximations formulas discussed in the text for representing the exact period. All error curves increases monotonically with θ_0. The horizontal dashed line marks the 0.1% level. The error committed in applying the small-angle approximation (RE_0) is greater than 0.1% for $\theta_0>7^{\circ}$ and reaches 15.3% for $\theta_0=90^{\circ}$. Note that the relative error committed by the proposed logarithmic formula ($RE_{\log}$) is smaller than that of the other simple approximation formulas, for all amplitudes.

Fig. 4. Comparison of the ratio T/T_0 for some approximation formulas and experimental data. The dotted curve is for the Bernoulli formula (see Eq. (10)). The dash-dotted curve is for the Kidd-Fogg formula (see Eq. (11)). The dashed line is for the Molina approximation formula (see Eq. (12)). The short-dashed line is for our logarithmic formula (see Eq. (9)). The solid line is the curve for the exact period, found via numerical integration of *K*(*k*) in Eq. (5). The experimental data were taken from Ref. 4 (+) and Ref. 17 (o). The solid (black) diamonds are our own experimental data. Note that the better agreement between the experimental data and the approximation curves is obtained for our logarithmic formula.

FIGURES

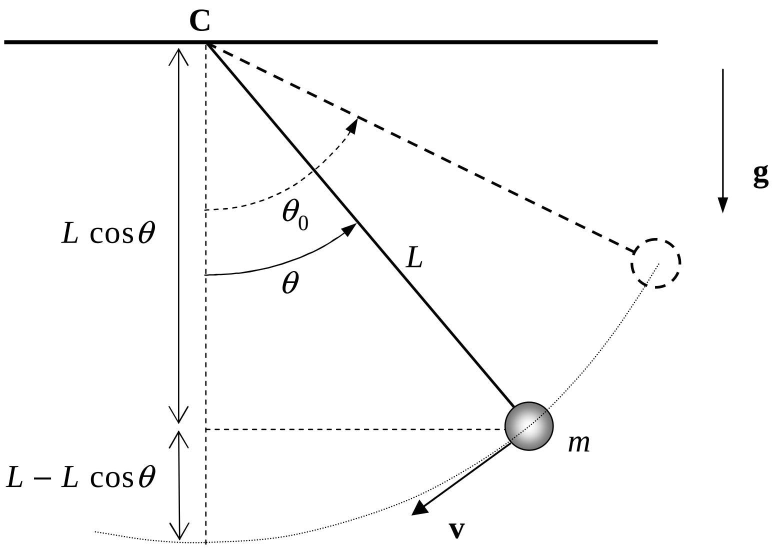


Fig. 1

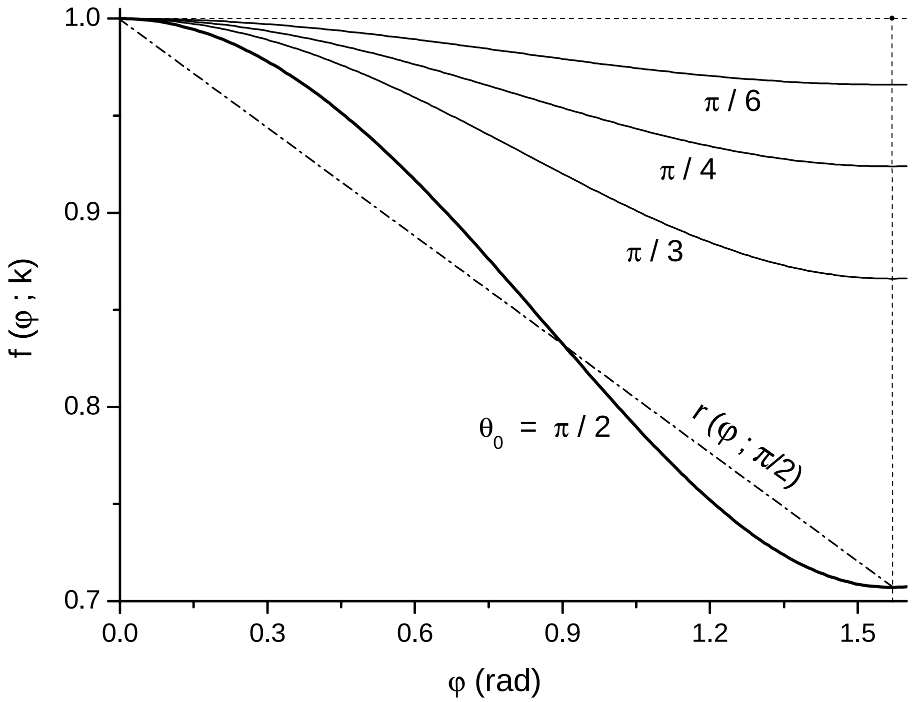
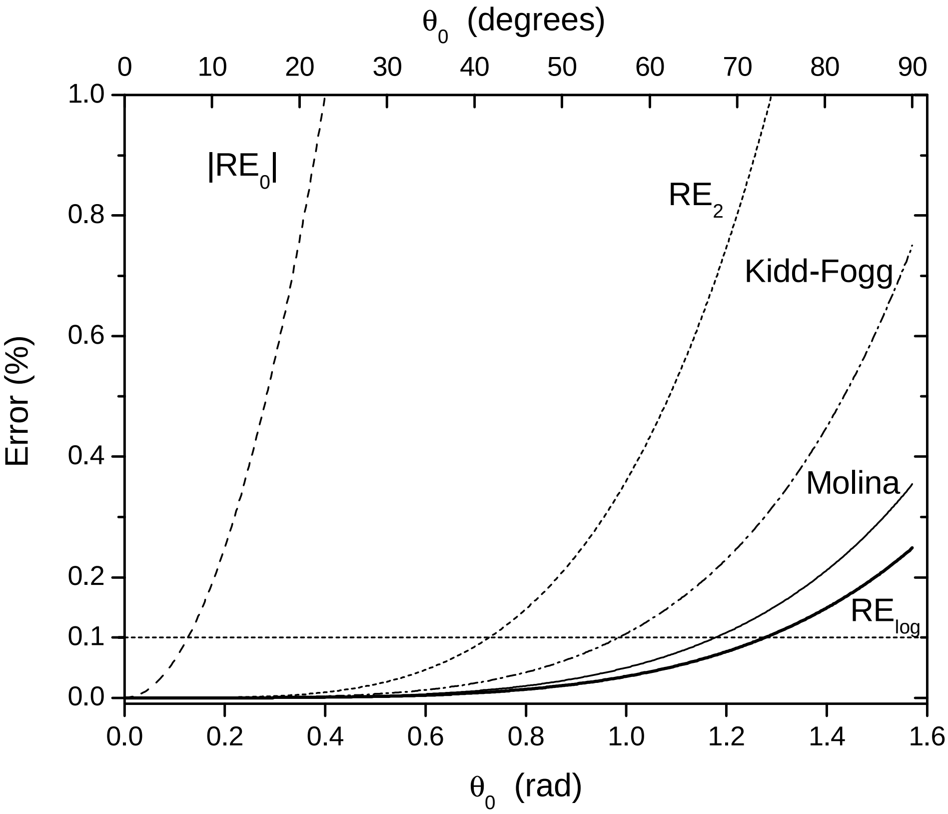
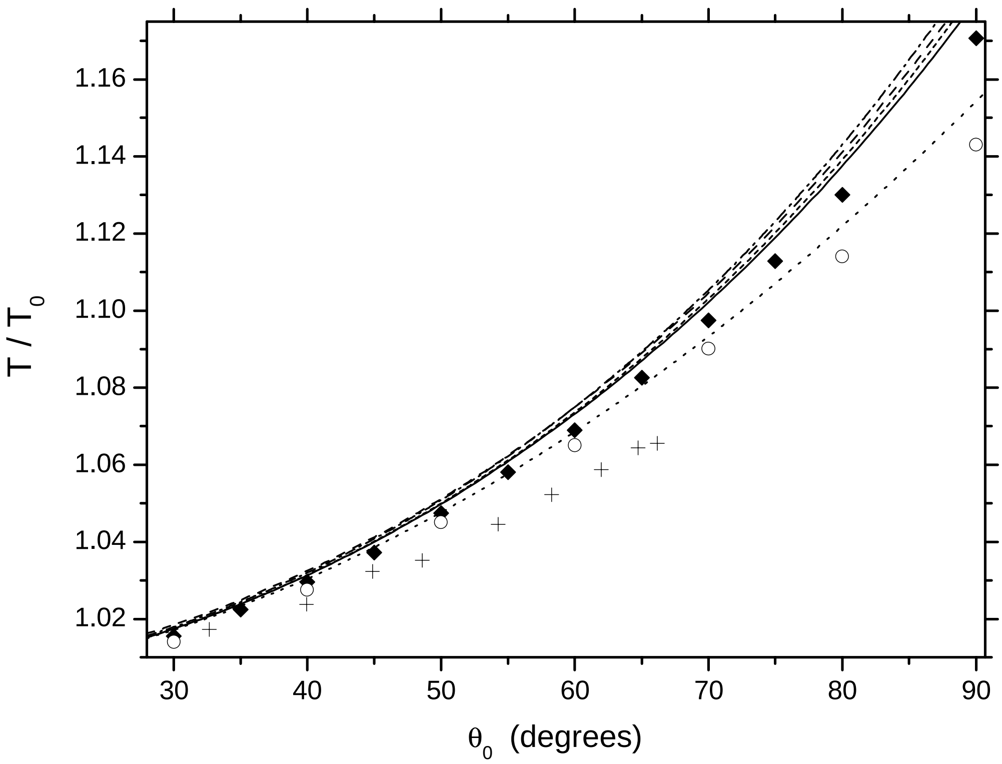


Fig. 2



θ0 (degrees)
0
10
20
30
40
50
60
70
80
90
1.0
0.8
0.6
0.4
0.2
0.1
0.0
Error (%)
|RE0|
RE2
Kidd-Fogg
Molina
RElog
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
θ0 (rad)

Fig. 3



1.16
1.14
1.12
1.10
1.08
1.06
1.04
1.02
T / T0
30
40
50
60
70
80
90
θ0 (degrees)

Fig. 4